

INTEGRATING RATIONAL FUNCTIONS

To find $\int \frac{N(x)}{D(x)} dx$, where $N(x)$ and $D(x)$ are both polynomials, and no cancellation is possible

Some rational integrands require partial fractions, others do not, and still others are a combination of the two. The process below outlines when you should and should not use partial fractions.

If degree of $N(x) \geq$ degree of $D(x)$ (ie. numerator has same degree as, or higher degree than, denominator)

1. Perform polynomial long division
2. Rewrite integrand as polynomial + remainder (with degree $<$ degree of $D(x)$)
3. Use processes below to find the integral of $\frac{\text{remainder}}{D(x)}$ (ie. set $N(x) = \text{remainder}$ and continue)

If degree of $D(x) = 1$ (ie. denominator is linear)

1. Let $u = D(x)$ & perform u - substitution
(or use guess & check - antiderivative is a multiple of $\ln|D(x)|$)

eg.
$$\int \frac{13}{7x-5} dx \quad \text{Let } u = 7x-5, \text{ so } \frac{du}{dx} = 7 \text{ and } dx = \frac{1}{7} du$$
$$= \int \frac{13 du}{7 u} = \frac{13}{7} \ln|u| + C = \frac{13}{7} \ln|7x-5| + C$$

If $N(x) = k \cdot D'(x)$ (ie. numerator is constant multiple of derivative of denominator)

TYPE 1

1. Let $u = D(x)$ & perform u - substitution
(or use guess & check - antiderivative is a multiple of $\ln|D(x)|$)

eg.
$$\int \frac{20-5x}{x^2-8x+25} dx \quad \text{Let } u = x^2-8x+25, \text{ so } \frac{du}{dx} = 2x-8 \text{ and } 20-5x = -\frac{5}{2}(2x-8)$$
$$= \int -\frac{5 du}{2 u} = -\frac{5}{2} \ln|u| + C = -\frac{5}{2} \ln|x^2-8x+25| + C$$

If degree of $D(x) = 2$ and is irreducible

(ie. denominator is quadratic with negative discriminant, so denominator has no real roots / only complex roots)

If $N(x) = c$ (ie. numerator is constant)

TYPE 2

1. Factor leading coefficient from $D(x)$ (ie. so denominator starts with x^2)
2. Complete the square for $D(x) = (x+h)^2 + a^2$
3. Factor a^2 from denominator, let $u = \frac{x+h}{a}$ & perform u - substitution

(or use $\int \frac{a}{(x+h)^2 + a^2} dx = \tan^{-1} \frac{x+h}{a}$)

eg.
$$\int \frac{7}{2x^2-16x+50} dx = \frac{7}{2} \int \frac{1}{x^2-8x+25} dx = \frac{7}{2} \frac{1}{3} \int \frac{3}{(x-4)^2+3^2} dx = \frac{7}{6} \tan^{-1} \frac{x-4}{3} + C$$

If $N(x) = ax + b$ (ie. numerator is linear)**TYPE 3**

- Use technique similar to partial fractions shortcut to rewrite numerator as $A \cdot D'(x) + B \cdot a$
(ie. constant multiple of derivative of denominator +
constant multiple of square root of “leftover” constant from completing square in denominator)
(see Partial Fractions Decomposition handout (special note regarding Math 1B))
- Split integrand into integrand of **TYPE 1** + integrand of **TYPE 2**
- Use processes above (NOTE: no absolute values required in $\ln(D(x))$)

eg.
$$\int \frac{3x-17}{x^2-8x+25} dx = \int \frac{A(2x-8)+B(3)}{(x-4)^2+3^2} dx \quad x=4: \quad 3(4)-17 = A(0)+B(3) \Rightarrow B = -\frac{5}{3}$$

coefficient of x : $3 = 2A \Rightarrow A = \frac{3}{2}$

$$= \int \frac{\frac{3}{2}(2x-8) - \frac{5}{3}(3)}{(x-4)^2+3^2} dx = \frac{3}{2} \int \frac{2x-8}{x^2-8x+25} dx - \frac{5}{3} \int \frac{3}{(x-4)^2+3^2} dx$$

$$= \frac{3}{2} \ln(x^2-8x+25) - \frac{5}{3} \tan^{-1} \frac{x-4}{3} + C$$

NOTE: TYPE 3 rational functions can also be integrated using a trigonometric substitution.

eg.
$$\int \frac{3x-17}{x^2-8x+25} dx = \int \frac{3x-17}{(x-4)^2+9} dx \quad \text{Let } x = 4 + 3 \tan \theta, \text{ so } dx = 3 \sec^2 \theta d\theta$$

$$= \int \frac{3(4+3 \tan \theta) - 17}{(3 \tan \theta)^2 + 9} 3 \sec^2 \theta d\theta = \int \frac{9 \tan \theta - 5}{3} d\theta = 3 \ln |\sec \theta| - \frac{5}{3} \theta + C$$

$$= 3 \ln \sqrt{(x-4)^2+9} - \frac{5}{3} \tan^{-1} \frac{x-4}{3} + C = \frac{3}{2} \ln((x-4)^2+9) - \frac{5}{3} \tan^{-1} \frac{x-4}{3} + C$$

All other cases require partial fractions

- Perform partial fractions decomposition (see handout)
NOTE: for irreducible quadratic denominators $d(x) = ax^2 + bx + c$
(or powers of these factors, ie. $[d(x)]^n$ or $(ax^2 + bx + c)^n$)
write numerator in $A \cdot D'(x) + B \cdot a$ form for **TYPE 3** to save work later on
- For all partial fractions with linear and irreducible quadratic denominators:
Use processes above
- For all partial fractions with denominator $[d(x)]^n = (ax + b)^n$ (ie. power of linear factor):
Let $u = ax + b$ & perform u - substitution
(or use guess & check – antiderivative is a multiple of $\frac{1}{(ax + b)^{n-1}}$)
- For all partial fractions with denominator $[d(x)]^n = (ax^2 + bx + c)^n$ (ie. power of irreducible quadratic factor):
Split integrand into integrand with numerator $A \cdot d'(x)$ + integrand with numerator B
For first integrand:
Let $u = ax^2 + bx + c$ & perform u - substitution
(or use guess & check – antiderivative is a multiple of $\frac{1}{(ax^2 + bx + c)^{n-1}}$)
For second integrand:
Factor leading coefficient from $ax^2 + bx + c$ (ie. so irreducible quadratic starts with x^2)
Complete the square for $x^2 + Bx + C = (x + h)^2 + k^2$
Let $x + h = k \tan \theta$ & perform trigonometric substitution
NOTE: This is the hardest type – there will be no required problems of this type on tests

Practice against the following examples:

TYPE 1

$$\int \frac{7}{3x+8} dx = \frac{7}{3} \ln|3x+8| + C$$

TYPE 1

$$\int \frac{6-9x}{3x^2-4x-4} dx = -\frac{3}{2} \ln|3x^2-4x-4| + C$$

TYPE 2

$$\int \frac{5}{4x^2+24x+52} dx = \frac{5}{4} \int \frac{1}{x^2+6x+13} dx = \frac{5}{4} \frac{1}{2} \int \frac{2}{(x+3)^2+2^2} dx = \frac{5}{8} \tan^{-1} \frac{x+3}{2} + C$$

TYPE 3

$$\begin{aligned} \int \frac{5x-7}{x^2+8x+25} dx &= \int \frac{\frac{5}{2}(2x+8) - 9(3)}{(x+4)^2+3^2} dx = \frac{5}{2} \int \frac{2x+8}{(x+4)^2+3^2} dx - 9 \int \frac{3}{(x+4)^2+3^2} dx \\ &= \frac{5}{2} \ln(x^2+8x+25) - 9 \tan^{-1} \frac{x+4}{3} + C \end{aligned}$$

MIXED

$$\begin{aligned} \int \frac{-20x-12}{(x+1)^2(x^2+4x+7)} dx &= \int \left(\frac{-6}{x+1} + \frac{2}{(x+1)^2} + \frac{3(2x+4) + \frac{4}{\sqrt{3}}(\sqrt{3})}{(x+2)^2+(\sqrt{3})^2} \right) dx \\ &= \int \frac{-6}{x+1} dx + \int \frac{2}{(x+1)^2} dx + 3 \int \frac{2x+4}{x^2+4x+7} dx + \frac{4}{\sqrt{3}} \int \frac{\sqrt{3}}{(x+2)^2+(\sqrt{3})^2} dx \\ &= -6 \ln|x+1| - \frac{2}{x+1} + 3 \ln(x^2+4x+7) + \frac{4}{\sqrt{3}} \tan^{-1} \frac{x+2}{\sqrt{3}} + C \end{aligned}$$

MIXED

$$\begin{aligned} \int \frac{3x^3-11x^2-49x}{x^2-5x-6} dx &= \int \left(3x+4 + \frac{-11x+24}{(x-6)(x+1)} \right) dx = \int \left(3x+4 - \frac{6}{x-6} - \frac{5}{x+1} \right) dx \\ &= \frac{3}{2} x^2 + 4x - 6 \ln|x-6| - 5 \ln|x+1| + C \end{aligned}$$